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A STUDY OF WAVE EQUATION BY SEPARATION OF VARIABLES

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ABSTRACT

Objective: In this paper we focus our study on wave equations. Westudying the solution of wave function by using separation of variables technique. The functions of several variablesand having worked through the concept of a partial derivative.

Materials and Methods: We first formulate the wave function u(x,t) where x is length of string. Solving the equation$u(x,t) = F(x)G(t)$ in two variables byusing the methods of Partial Differential Equation. We get the following equation

$$\frac{G''(t)}{c^2G(t)} = \frac{F''(x)}{F(x)} = k$$

where k is constant.

Results: We are going to check the possible for the constant k in the above equation. First we consider k = 0 then we get u(x, t)= (px+r) (at+b) where a,b, p and r, were constants.

Secondly we consider k> 0 then we get $F(x) = Ae^{\omega x} + Be^{-\omega x}$ whereA and B are constants and $\omega = \sqrt{k}$.

Lastly we consider k< 0 then we get $u_n(x,t) = F(x)G(t) = \left(C\cos(\lambda_n t) + D\sin(\lambda_n t)\right)\sin\left(\frac{n\pi}{l}x\right)$ where the integer n that was used isidentified by the subscript in $u_n(x,t)$ and λ_n , and arbitrary constants are C and D.

Conclusion: The solutions given in the first two cases are dull solutions.The solution given in the last case really does satisfy the wave equation. We can find a particular solution function for varying values of time, t,

Keywords: - Wave equation, Partial derivatives, Exponential functions, Frequency, Poisson equation.

INTRODUCTION

We consider a close look at the PDEs. Try to classify using the given terminology. Note that the f(x,y) function in the Poisson equation is just a function of the variables x and y, it has nothing to do with u(x,y). To solve Partial Differential Equations is considerably more difficult in general than to solve Ordinary Differential Equations, as the complications involved can be great.The wave equations can be solved by several approaches. The first one will using a technique called separation of variables.[1] The second technique, used is a transformation trick that also reduces the complexity of the original PDE, but in a very different manner.

The advantage of an abstract approach is that it concentrates on the required facts, so that these facts become clearly visible and one's attention is not disturbed by non important details. Moreover, by developing a box of tools in the abstract framework, one is equipped for solving many different problems. In the abstract approach, we can usually starts from a set of elements satisfying certain axioms. The theory then consists of logical consequences which are derived from the axioms and are derived as theorems once and for all. These general theorems can then later be applied to various concrete special sets satisfying these axioms.

We will develop such an abstract scheme for doing calculus in function spaces and other infinite-dimensional spaces, and this is what this course is about.[2]We will be equipped with a set of tools for solving these problems, and in particular, we will return to the optimal mining operation problem again and solve it.

MATERIALS AND METHODS

First, note that for a particular wave equation situation, in addition to the Partial Differential Equation, we will also consider boundary conditions arising from the fact that the endpoints of the string are attached solidly, x = 0 at the left end of the string.At the other end of the string, we suppose has overall length l. Let's start the process of solving the Partial Differential Equation by first figuring out what these boundary conditions imply for the solution function i.e. $u(x,t)$.

(1) $u(0,t) = 0$ and $u(l,t) = 0$

for all values of t are the boundary conditions for this wave equation [5]. These will be key when we later on need to sort through probable solution functions for functions that will satisfy our particular vibrating string set-up.

Note that we probably need to specify what the shape of the string is right when time t = 0, and you are right to come up with a particular solution function, we wantto know $u(x,0)$. In fact we will also need to know the initial velocity of the stringi. e. $u_t(x,0)$.

These two requirements are called the initial conditions for this wave equation, and are also necessary to specify a specific vibrating string solution. For instance, as the simplest example of initial conditions, if no one is plucking the string, and it's perfectly flat to start with, then the initial conditions will be $u(x,0) = 0$ (for perfectly flat string) with initial velocity, $u_t(x,0) = 0$ [5,6] Here, the solution function is pretty unenlightening, it's just $u(x,t) = 0$ when no movement of the string through time.

To use the separation of variables technique we make the key assumption that whatever the solution function is, that it can be written as the multiplication of two independent functions, each one of which depends on just one of these two variables, x or t. Thus, imagine that the solution function, $u(x,t)$ can be written as

(2) $$u(x,t) = F(x)G(t)$$

where F& G are single variable functions of x and t respectively.[2] Differentiating this equation for $u(x,t)$ two times with respect to each variable yields

(3) $$\frac{\partial^2 u}{\partial x^2} = F''(x)G(t)$$ and
$$\frac{\partial^2 u}{\partial t^2} = F(x)G''(t)$$

Thus when we substitute these two equations back into the original wave equation, which is

(4) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

then we get that

(5)

$$\frac{\partial^2 u}{\partial t^2} = F(x)G''(t) = c^2 \frac{\partial^2 u}{\partial x^2} = c^2 F''(x)G(t)$$

Here is where the separation of variables assumption pays off, because now if we separate these equation above so that the terms involving F and its second derivative are on one side, and likewise the terms involving G and its derivative are on the other, we get that

(6) $$\frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)}$$

We have an equality where the left-hand side just depends on the variable t, and the right-hand side just depends on x[5,6]. Here comes the critical observation how can two functions, one just depending on t, and one just on x, be equal for all possible values of t and x? The answer is that they must each be constant, for otherwise the equality could not possibly hold for all possible combinations of t and x. Thus we get

(7) $$\frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = k$$

where k is a constant. First we will check the possible cases for k.

RESULTS

Case One: k = 0

Suppose k equals 0. Then the equations in (7) can be written as

(8) $$G''(t) = 0 \cdot c^2 G(t) = 0$$ and
$$F''(x) = 0 \cdot F(x) = 0$$

which yields with very little effort two solution functions for F and G:

(9) $$G(t) = at + b$$ and $$F(x) = px + r$$

where a,b, p and r, were constants (note how easy it is to solve such simple Ordinary Differential Equations versus trying to deal with two variables at once, hence the power of the separation of variables approaches) [2,10].

Putting these back together to form $u(x,t) = F(x)G(t)$, then the next thing we need to do is to note what the boundary conditions from equation (1) force upon us i. e.

(10) $$u(0,t) = F(0)G(t) = 0$$ and
$$u(l,t) = F(l)G(t) = 0$$ for all values of t

Unless $G(t) = 0$ (which would then mean that $u(x,t) = 0$, giving us the very dull solution equivalent to a flat, un plucked string then this will imply that

(11) $$F(0) = F(l) = 0.$$

But how can a linear function have two roots? Only by being identically equal to 0, thus it must be the case that $F(x) = 0$. then we still get that $u(x,t) = 0$, and we end up with the dull solution again, the only possible solution if we starting with k = 0.

Case Two: k > 0

Consider, if k is positive, then from equation (7) we again start with the equation

(12) $$G''(t) = kc^2 G(t)$$

and

(13) $$F''(x) = kF(x)$$

We are looking for functions whose second derivatives give back the original function, multiplied by a positive constant[7,8]. Possible candidate solutions to consider include the exponential, sine and cosine functions. These sine and cosine functions don't work here, as their second derivatives are negative the original function, so we are left with the exponential functions.

Let's take a look at equation (13) more closely first, as we already know that the boundary conditions implies conditions specifically for $F(x)$, Then the conditions in (11). Solutions for $F(x)$ include anything of the form

(14) $$F(x) = Ae^{\omega x}$$

where $\omega^2 = k$ and A is a constant. Since ω could be positive or negative, and since solutions to equation (13) can be added together to form more solutions (note (13) is an example of a second order linear homogeneous ordinary differential equation, so that the superposition principle holds, then the general solution for equation (13) is as below,

(14) $$F(x) = Ae^{\omega x} + Be^{-\omega x}$$

where A and B are constants and $\omega = \sqrt{k}$. Knowing that $F(0) = F(l) = 0$, then unfortunately the only possible values of A and B that work are $A = B = 0$, i.e. that $F(x) = 0$. Thus, once again we end up with $u(x,t) = F(x)G(t) = 0 \cdot G(t) = 0$, the dull solution which gets once more.

Case Three: k < 0

Lastly we considering the negative values for k, So we go back to equations (12) and (13) again, but now working with k as a negative constant. So, again we have these equations

(12) $$G''(t) = kc^2 G(t)$$ and

(13) $$F''(x) = kF(x)$$

Exponential functions won't satisfy these two ordinary differential equations, but now the sine and cosine functions will be used [9]. The general solution function for (13) is now

(15) $$F(x) = A\cos(\omega x) + B\sin(\omega x)$$

where again A and B are constants and now we have $\omega^2 = -k$. Again, we consider the boundary conditions that specified that $F(0) = F(l) = 0$. Substituting in 0 for x in (15) leads to

$$F(0) = A\cos(0) + B\sin(0) = A = 0 \tag{16}$$

Therefore $F(x) = B\sin(\omega x)$. Next, we consider $F(l) = B\sin(\omega l) = 0$.

We can assume that B isn't equal to 0, otherwise $F(x) = 0$ which would mean that $u(x,t) = F(x)G(t) = 0 \cdot G(t) = 0$, again, the trivial unplucked string solution [7].

With $B \neq 0$, then it must be the case that $\sin(\omega l) = 0$ in order to have $B\sin(\omega l) = 0$. The only way that this can happen is for ωl to be a multiple of π. Then we get that

(17) $\omega l = n\pi$ or $\omega = \dfrac{n\pi}{l}$ (where n is an integer) This means that there is an infinite set of solutions to consider (letting the constant B be equal to 1 for now), one for each possible integer n.

$$F(x) = \sin\left(\frac{n\pi}{l}x\right) \tag{18}$$

Well, we would be done at this point, except that the solution function $u(x,t) = F(x)G(t)$ and we've neglected to figure out what the other function, $G(t)$, equals. now, we return to the ordinary differential equation in equation (12):

$$G''(t) = kc^2 G(t) \tag{12}$$

where, again, we are working with k, a negative number. From the solution for $F(x)$ we have determined that the only possible values that end up leading to non-trivial solutions are with the constant

$k = -\omega^2 = -\left(\dfrac{n\pi}{l}\right)^2$ for n some integer. Again, we get an infinite set of solutions for (12) that can be written in the form

$$G(t) = C\cos(\lambda_n t) + D\sin(\lambda_n t) \tag{19}$$

where C and D are constants and $\lambda_n = c\sqrt{-k} = c\omega = \dfrac{cn\pi}{l}$, where n is the same integer that showed up in the solution for $F(x)$ in (18) (we're labeling λ with a subscript "n" to identify which value of n is used [8,9].

Now we find for all we have to do is to drop our solutions for $F(x)$ and $G(t)$ into $u(x,t) = F(x)G(t)$, and the result is

$$u_n(x,t) = F(x)G(t) = \left(C\cos(\lambda_n t) + D\sin(\lambda_n t)\right)\sin\left(\frac{n\pi}{l}x\right) \tag{20}$$

where the integer n that was used is identified by the subscript in $u_n(x,t)$ and λ_n, and arbitrary constants are C and D.

At this point you should be in the habit of immediately checking solutions to these differential equations. Is (20) really a solution for the original wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

and does it actually satisfy the boundary conditions $u(0,t) = 0$ and $u(l,t) = 0$ for all values of t.

CONCLUSION

The solution given in the last section really does satisfy the one-dimensional wave equation. To think about what the solutions look like, you could graph a particular solution function for varying values of time, t, and then examine how the string vibrates over time for solution functions with different values of n and constants C and D. However, as the functions involved are fairly simple, it's possible to make sense of the solution $u_n(x,t)$ functions with just a little more effort.

For instance, over time, we can see that the $G(t) = \left(C\cos(\lambda_n t) + D\sin(\lambda_n t)\right)$ part of the function is periodic with period equal to $\dfrac{2\pi}{\lambda_n}$. This means that it has a frequency equal to $\dfrac{\lambda_n}{2\pi}$ cycles per unit time.

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